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# Points and topologies in rigid geometry

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## 1 Introduction

This paper has its origin in notes of the second author and remarks in [P]. Its purpose is to clarify the various concepts of points of an affinoid space and topologies on sets of points which can be found in [B, D, H, P, S].

The set of ordinary points of an affinoid space  $X$  over a complete non-archimedean valued field  $k$  is too small for the collection of abelian sheaves on  $X$ . The family of stalks in the ordinary points does not detect the vanishing of a sheaf. One introduces the notion of a prime filter  $p$  on  $X$  to remedy this. The collection of all prime filters is called  $\mathcal{P}(X)$ . For every  $p \in \mathcal{P}(X)$  and every abelian sheaf  $F$  on  $X$  one can form a stalk  $F_p$  which is an abelian group. The functor  $F \mapsto F_p$  is exact and  $F = 0$  if all  $F_p$  are zero.

The set  $\mathcal{P}(X)$  has a natural topology on it. For this topology  $\mathcal{P}(X)$  is quasi-compact but not Hausdorff. The topological space  $\mathcal{P}(X)$  can be identified with the projective limit of a certain family of schemes of finite type over the residue field of  $k$ . For a field  $k$  with a discrete valuation this family consists of the special fibers of all formal schemes  $\mathcal{X}$  of finite type and flat over the valuation ring  $k^0$  of  $k$  such that the “generic fiber”  $\mathcal{X} \otimes k$  is isomorphic to  $X$ .

The set  $\mathcal{P}(X)$  is shown to be isomorphic to a collection of valuations  $\text{Val}(X)$  on the affinoid algebra  $\mathcal{O}(X)$  of  $X$ . The latter space is the building block in Huber’s approach to analytic spaces. The valuations of (real) rank 1 correspond to a certain subset  $\mathcal{M}(X)$  of  $\mathcal{P}(X)$ . This subset consists of the maximal filters on  $X$ . There is a natural retraction map  $r: \mathcal{P}(X) \rightarrow \mathcal{M}(X)$ . Let  $\mathcal{M}(X)_{qt}$  denote the set  $\mathcal{M}(X)$  provided with the quotient topology. It turns out to be a compact Hausdorff space. The set  $\mathcal{M}(X)$  plays a central role in Berkovich theory of analytic spaces. He provides  $\mathcal{M}(X)$  with a certain topology which we will show to coincide with the quotient topology. This answers a question in [D].

The category of abelian sheaves on  $X$  turns out to be equivalent to the category of abelian sheaves on the topological space  $\mathcal{P}(X)$ . Using the retraction map  $r : \mathcal{P}(X) \rightarrow \mathcal{M}(X)$  and the nice topological structure of  $\mathcal{M}(X)_{qt}$  one finds that the category of overconvergent sheaves on  $X$  coincides with the category of abelian sheaves on  $\mathcal{M}(X)_{qt}$ . This fact was already proven in [S] and led to an easy proof of a base change theorem for rigid analytic spaces.

For part of our results about the spaces  $\mathcal{P}(X)$  and  $\text{Val}(X)$  one can find brief indications already in [H].

In the last section we extend our results to general rigid spaces.

## 2 Points and valuations

Let  $X$  be an affinoid space over a complete non-archimedean valued field  $k$ . We will use the notations  $k^0 := \{\lambda \in k : |\lambda| \leq 1\}$  and  $\bar{k}$  for the residue field of  $k$ . Further  $\pi$  denotes some element in  $k$  with  $0 < |\pi| < 1$ . The algebra of functions on  $X$  will be denoted by  $\mathcal{O}(X)$ . This algebra has a presentation  $\alpha : k\langle T_1, \dots, T_m \rangle \rightarrow \mathcal{O}(X)$  which induces a norm  $|\cdot|_\alpha$  on  $\mathcal{O}(X)$  (compare [BGR] 6.1.1). The subring of elements with norm  $\leq 1$  is denoted by  $\mathcal{O}(X)_\alpha^0$ . The subring of the elements with spectral (semi-)norm  $\leq 1$  ([BGR] 6.2.1.2) is denoted by  $\mathcal{O}(X)^0$ . One defines  $\mathcal{O}(X)^{00} \subseteq \mathcal{O}(X)^0$  to consist of the elements  $f$  such that  $\lim_{n \rightarrow \infty} |f^n|_\alpha = 0$ . An  $f \in \mathcal{O}(X)$  belongs to  $\mathcal{O}(X)^{00}$  if and only if its spectral (semi-)norm is  $< 1$  ([BGR] 6.2.3.2). The ring  $\mathcal{O}(X)^0 / \mathcal{O}(X)^{00}$  is a finitely generated reduced  $\bar{k}$ -algebra and is called the *canonical reduction* of  $\mathcal{O}(X)$ . The corresponding affine scheme over  $\bar{k}$  is denoted by  $\bar{X}$  and its set of closed points by  $\bar{X}_{cl}$ . The dimension of  $\bar{X}$  is equal to the dimension of  $X$ .

The space  $X$  as a set consists of the maximal ideals in  $\mathcal{O}(X)$ . The finite unions of affinoid subdomains in  $X$  are called special subsets. We always give  $X$  the following Grothendieck topology:

(1) The admissible subsets of  $X$  are the special subsets.

(2) For a special subset  $U$  the family  $\text{Cov}(U)$  consists of the coverings by special subsets which have a finite subcovering.

A rational set  $R(f_0, \dots, f_n)$  in  $X$  is given by elements  $f_0, \dots, f_n \in \mathcal{O}(X)$  generating the unit ideal and is defined to be

$$R(f_0, \dots, f_n) := R_X(f_0, \dots, f_n) \\ = \{x \in X : |f_0(x)| \geq |f_i(x)| \text{ for all } 1 \leq i \leq n\}.$$

For any rational set  $U \subseteq X$  the coverings of the form

$$U = \bigcup_{0 \leq i \leq n} R_U(f_i, f_0, \dots, f_{i-1}, f_{i+1}, \dots, f_n)$$

are called *rational coverings*. The rational sets together with their rational coverings generate the Grothendieck topology of  $X$  ([BGR] 8.2.2.2). This has two simple consequences for abelian sheaves on  $X$ . Firstly a sheaf is determined by its sections in rational sets. Secondly let  $P$  be a presheaf on  $X$  and let  $F$

be its sheafification; if  $P$  satisfies the sheaf axiom for rational coverings then we have  $P(U) = F(U)$  for any rational set  $U$ .

A *filter*  $f$  on  $X$  is a collection of special subsets of  $X$  satisfying:

(p1)  $X \in f$  and  $\emptyset \notin f$ ;

(p2) if  $U_1, U_2 \in f$  then  $U_1 \cap U_2 \in f$ ;

(p3) if  $U \in f$  and the special subset  $V$  contains  $U$  then  $V \in f$ .

A *prime filter*  $p$  is a filter which in addition fulfills:

(p4) If  $U \in p$  and  $U = U_1 \cup U_2$  with special subsets  $U_i$  then  $U_1$  or  $U_2 \in p$ .

This is, of course, equivalent to the condition:

(p4)' If  $U = \bigcup_{i \in I} U_i$  is an admissible covering of  $U \in p$  then  $U_{i_0} \in p$  for some  $i_0 \in I$ .

Let  $\mathcal{P}(X)$  denote the set of all prime filters on  $X$ . The filters on  $X$  are ordered with respect to inclusion. Let  $\mathcal{M}(X)$  be the set of maximal filters. We have  $\mathcal{M}(X) \subseteq \mathcal{P}(X)$ . This is a special case of the following basic argument which later on will be used several times.

*Remark 1.* Let  $s$  be any family of special subsets of  $X$  which is closed with respect to finite unions. Assume that there is a filter  $f$  on  $X$  such that  $f \cap s = \emptyset$ . Then there is a filter  $p$  on  $X$  containing  $f$  which is maximal with respect to  $p \cap s = \emptyset$  and any such filter is a prime filter.

*Proof.* The existence of  $p$  follows by Zorn's lemma. It remains to verify (p4) for  $p$ . Let  $U, U_1, U_2$  be special subsets in  $X$  such that  $U \in p$  and  $U = U_1 \cup U_2$ . If  $U_1 \notin p$  then the collection  $p_1$  of special subsets of  $X$  defined by  $V \in p_1$  if  $V$  contains some  $W \cap U_1$  with  $W \in p$  still satisfies  $X \in p_1$ , (p2), and (p3). Since  $p_1$  is larger than  $p$  we must have, by the maximality of  $p$ , that  $p_1$  contains a member of  $s \cup \{\emptyset\}$ . Assume now that neither  $U_1$  nor  $U_2$  belong to  $p$ . Then there are  $V_i \in s \cup \{\emptyset\}$  and  $W_i \in p$  such that  $W_i \cap U_i \subseteq V_i$ . Hence  $W := W_1 \cap W_2 \in p$  satisfies  $W \cap U \subseteq V_1 \cup V_2 \in s \cup \{\emptyset\}$ . This is a contradiction.

A *valuation*  $(\mathfrak{p}, A)$  on  $X$  is a pair consisting of a prime ideal  $\mathfrak{p} \subseteq \mathcal{O}(X)$  and a valuation ring  $A$  in the field of fractions of  $\mathcal{O}(X)/\mathfrak{p}$  such that:

(v1)  $\mathcal{O}(X)^0 + \mathfrak{p}/\mathfrak{p} \subseteq A$ ;

(v2) the intersection of all  $\pi^n A$  is 0.

Usually we will denote by  $\phi$  the residue class map from  $\mathcal{O}(X)$  into the field of fractions  $K$  of  $\mathcal{O}(X)/\mathfrak{p}$ . Let  $\text{Val}(X)$  be the set of all valuations on  $X$ .

We note that one can replace (v1) by the weaker condition

(v1)'  $\phi(\mathcal{O}(X)^0_{\alpha}) \subseteq A$ .

Indeed, by Noether normalization we have a finite monomorphism  $k\langle T_1, \dots, T_d \rangle \hookrightarrow \mathcal{O}(X)$  such that the  $T_i$  are mapped into  $\mathcal{O}(X)^0_{\alpha}$ . An element  $f \in \mathcal{O}(X)$  with spectral (semi-)norm  $\leq 1$  is integral over  $k^0\langle T_1, \dots, T_d \rangle$  ([BGR] 6.3.4.1). This last ring is mapped by  $\phi$  into  $A$ . Hence  $\phi(f)$  is integral over  $A$  and therefore lies in  $A$ .

Also note that (v2) means that the valuation topology on  $A$  is the  $\pi$ -adic topology. It is sometimes convenient to replace  $A$  by its completion  $\hat{A}$ , the projective limit of the  $A/\pi^n A$ . It follows from (v2) that  $A$  is a subring of  $\hat{A}$ .

One easily verifies that  $\hat{A}$  has no zero divisors and is in fact a valuation ring. The field of fractions of  $\hat{A}$  will be denoted by  $\hat{K}$ . Let  $\mathfrak{m}_A$  be the maximal ideal in  $A$ . The map  $A \rightarrow A/\mathfrak{m}_A$  extends to a map  $\hat{A} \rightarrow A/\mathfrak{m}_A$  and therefore the maximal ideal  $\mathfrak{m}_{\hat{A}}$  contains  $\mathfrak{m}_A$ . Suppose now that  $a \in K \setminus A$ . Then  $a^{-1} \in \mathfrak{m}_A$ . Hence  $a^{-1} \in \mathfrak{m}_{\hat{A}}$  and  $a \notin \hat{A}$ . This shows that  $\hat{A} \cap K = A$ . More general, for any  $a \in K$  one has  $a\hat{A} \cap K = aA$ .

The aim of this section is to find a natural bijection  $\text{Val}(X) \rightarrow \mathcal{P}(X)$ . Let a valuation  $(\mathfrak{p}, A)$  be given. Define a family  $p = p(\mathfrak{p}, A)$  of special subsets of  $X$  as follows:

$U$  belongs to  $p$  if it contains a rational set  $R(f_0, \dots, f_n)$  such that  $\phi(f_i) \in \phi(f_0)A$  for all  $1 \leq i \leq n$ .

**Proposition 2.** i) If  $R(f_0, \dots, f_n) \subseteq R(g_0, \dots, g_m)$  and  $\phi(f_i) \in \phi(f_0)A$  for all  $i$  then  $\phi(g_j) \in \phi(g_0)A$  for all  $j$ ;  
ii)  $p$  is a prime filter on  $X$ .

*Proof.* i. The homomorphism  $\phi : \mathcal{O}(X)_\alpha^0 \rightarrow A$  extends to the homomorphism

$$\begin{aligned} \phi_1 : \mathcal{O}(X)_\alpha^0[T_1, \dots, T_n] &\rightarrow A \\ T_i &\mapsto \frac{\phi(f_i)}{\phi(f_0)}. \end{aligned}$$

By taking limits  $\phi_1$  extends uniquely to a homomorphism  $\phi_2 : \mathcal{O}(X)_\alpha^0(T_1, \dots, T_n) \rightarrow \hat{A}$ . The corresponding homomorphism  $\mathcal{O}(X)(T_1, \dots, T_n) \rightarrow \hat{K}$  factors over  $\phi_3 : R := \mathcal{O}(X)(T_1, \dots, T_n)/(f_1 - f_0 T_0, \dots, f_n - f_0 T_n) \rightarrow \hat{K}$ . We have  $R = \mathcal{O}(R(f_0, \dots, f_n))$ . Let us note here that since the image of  $\phi_3$  is not  $\{0\}$  it follows that  $R(f_0, \dots, f_n)$  is not empty. By construction  $\phi_3(R_\beta^0) \subseteq \hat{A}$  where  $|\cdot|_\beta$  is the norm on  $R$  induced by the given presentations of  $R$  and  $\mathcal{O}(X)$ . As noted before, this implies that also  $\phi_3(R^0) \subseteq \hat{A}$ .

We consider now the restriction map  $\mathcal{O}(R(g_0, \dots, g_m)) \rightarrow \mathcal{O}(R(f_0, \dots, f_n))$ . The images of the elements  $\frac{g_j}{g_0}$  have spectral (semi-)norm  $\leq 1$  in  $R$ . Hence

$$\frac{\phi(g_j)}{\phi(g_0)} = \phi_3 \left( \frac{g_j}{g_0} \right) \in \hat{A} \cap K = A.$$

ii. We have already seen that the empty set does not belong to  $p$ . Further  $X = R(1, 0)$  obviously belongs to  $p$ . This proves (p1). The formula  $R(f_0, \dots, f_n) \cap R(g_0, \dots, g_m) = R(\dots, f_i g_j, \dots)$  proves (p2). The condition (p3) holds by construction. For (p4) we first consider the case of a rational covering of  $X$  given by  $g_0, \dots, g_m \in \mathcal{O}(X)$  generating the unit ideal. The fractional ideal  $\phi(g_0)A + \dots + \phi(g_m)A$  in  $K$  is generated by some  $\phi(g_{i_0})$ . Hence  $R(g_{i_0}, g_0, \dots, g_m) \in p$ . In order to prove (p4) in general it suffices, by the definition of  $p$ , to consider a rational set  $U = R(f_0, \dots, f_m)$ . Because  $p$  is already known to be a filter (p4) for  $U$  is a consequence of (p4) for  $X$  by the following fact: For any admissible covering of  $U$  there exists a rational covering of

$X$  whose restriction to  $U$  is finer than the given covering. This is well-known but for the convenience of the reader we include the argument.

First we refine the given covering into a rational covering

$$U = \bigcup_{0 \leq j \leq m} U_j \quad \text{where } U_j = R_U(g_j, g_0, \dots, g_{j-1}, g_{j+1}, \dots, g_m)$$

with  $g_0, \dots, g_m \in \mathcal{O}(U)$  generating the unit ideal. Since  $g_j$  is invertible on  $U_j$  we find an  $\varepsilon > 0$  such that

$$|g_j(x)| > \varepsilon \quad \text{for any } x \in U_j \text{ and any } 0 \leq j \leq m.$$

Now we choose elements  $g'_0, \dots, g'_m \in \mathcal{O}(X)[f_0^{-1}]$  such that

$$|g_j(x) - g'_j(x)| < \varepsilon \quad \text{for any } x \in U \text{ and any } 0 \leq j \leq m.$$

Then the  $g'_j$  cannot have a common zero in  $U$  and therefore generate the unit ideal in  $\mathcal{O}(U)$ ; also by construction we have

$$U_j = R_U(g'_j, g'_0, \dots, g'_{j-1}, g'_{j+1}, \dots, g'_m) \quad \text{for any } 0 \leq j \leq m.$$

Moreover we may multiply all the  $g'_j$  by an appropriate power of  $f_0$  without changing the  $U_j$ . This shows that from the start we may assume without changing the given rational covering that the  $g_0, \dots, g_m$  extend to functions on  $X$ . In this situation we choose a  $\lambda \in k^\times$  such that  $|\lambda| < \varepsilon$ . The rational covering of  $X$  given by the functions  $\lambda, g_0, \dots, g_m$  then has the wanted property.

We next construct a map  $\mathcal{P}(X) \rightarrow \text{Val}(X)$ . Let  $p$  be a prime filter on  $X$ . We put

$$\begin{aligned} \mathcal{O}_p &:= \varinjlim_{U \in p} \mathcal{O}(U) = \varinjlim \{ \mathcal{O}(U) : U \in p \text{ rational} \} \quad \text{and} \\ \mathcal{O}_p^0 &:= \varinjlim_{U \in p} \mathcal{O}^0(U) = \varinjlim \{ \mathcal{O}^0(U) : U \in p \text{ rational} \}. \end{aligned}$$

Define

$$\|f\|_p := \inf \|f\|_U \quad \text{for } f \in \mathcal{O}_p;$$

here  $\|\cdot\|_U$  denotes the spectral (semi-)norm on  $U$  and the infimum is taken over all  $U \in p$  such that  $f$  is defined on  $U$ . Let

$$\mathfrak{m}_p := \{f \in \mathcal{O}_p : \|f\|_p = 0\}.$$

Clearly  $\mathfrak{m}_p$  is an ideal in both rings  $\mathcal{O}_p$  and  $\mathcal{O}_p^0$ . Put

$$k_p := \mathcal{O}_p / \mathfrak{m}_p \quad \text{and} \quad k_p^0 := \mathcal{O}_p^0 / \mathfrak{m}_p.$$

**Proposition 3.** i.  $\mathfrak{m}_p$  is the unique maximal ideal of  $\mathcal{O}_p$ ;

ii.  $k_p^0$  is a valuation ring with field of fractions  $k_p$ ;

iii. let  $\mathfrak{p}$  be the kernel of the homomorphism  $\mathcal{O}(X) \rightarrow k_p$  and let  $A$  be the preimage of  $k_p^0$  in the field of fractions of  $\mathcal{O}(X)/\mathfrak{p}$ ; then  $\text{val}(p) := (\mathfrak{p}, A)$  is a valuation on  $X$ .

*Proof.* i. Let  $f \in \mathcal{O}_p$  be defined on some  $U \in p$ . If for some  $\varepsilon \in |k|$ ,  $\varepsilon > 0$ , the set  $\{x \in U : |f(x)| \geq \varepsilon\}$  belongs to  $p$  then  $f$  is an invertible element of  $\mathcal{O}_p$ . If not then  $\{x \in U : |f(x)| \leq \varepsilon\} \in p$  for all  $\varepsilon$  and so  $f \in \mathfrak{m}_p$ .

ii. We will not distinguish in notation between  $f \in \mathcal{O}_p$  and its image in  $k_p$ . Suppose that  $f \in k_p$  does not lie in  $k_p^0$ . Let  $f$  be defined on some  $U \in p$ . Then  $\{x \in U : |f(x)| \leq 1\}$  does not belong to  $p$ . Hence the set  $\{x \in U : |f(x)| \geq 1\} \in p$  and so  $f^{-1} \in k_p^0$ .

iii. The only non-trivial item to verify is (v2). But  $\pi^n A \subseteq \pi^n k_p^0$ . Let  $f \in \mathcal{O}_p$  represent an element in the intersection of all  $\pi^n k_p^0$ . Then clearly  $\|f\|_p = 0$  and the image of  $f$  in  $k_p^0$  is 0.

**Theorem 4.** *The maps  $p(\cdot)$  and  $\text{val}(\cdot)$  between  $\text{Val}(X)$  and  $\mathcal{P}(X)$  are each others inverses.*

*Proof.* Let  $(\mathfrak{p}, A)$  be given and let  $p := p(\mathfrak{p}, A)$ . In the proof of Proposition 2 we have seen that, for any  $U = R(f_0, \dots, f_n) \in p$ , there is a unique continuous extension  $\phi_U : \mathcal{O}(U) \rightarrow \hat{K}$  of  $\phi$  satisfying  $\phi_U(\mathcal{O}(U)^0) \subseteq \hat{A}$ . This induces a homomorphism  $\phi_p : \mathcal{O}_p \rightarrow \hat{K}$  such that  $\phi_p(\mathcal{O}_p^0) \subseteq \hat{A}$ . For  $f \in \mathfrak{m}_p$  one has

$$f \in \bigcap_{n \in \mathbb{N}} \pi^n \mathcal{O}_p^0 \quad \text{and so} \quad \phi_p(f) \in \bigcap_{n \in \mathbb{N}} \pi^n \hat{A} = \{0\}.$$

Hence  $\phi_p$  induces an injection  $k_p \subseteq \hat{K}$  such that  $k_p^0 \subseteq \hat{A}$ . Let  $f \in k_p \cap \hat{A}$ . One can represent  $f$  by some element  $g \in \mathcal{O}(U)$  with  $U \in p$ . The condition  $\phi_U(g) \in \hat{A}$  implies that  $\{x \in U : |g(x)| \leq 1\} \in p$  and therefore  $f \in k_p^0$ . Hence  $k_p^0 = k_p \cap \hat{A}$ . This implies  $k_p^0 \cap K = A$  and we have proved that  $\text{val}(p) = (\mathfrak{p}, A)$ . Let a prime filter  $p$  be given and let  $(\mathfrak{p}, A) := \text{val}(p)$ . The filter induces a homomorphism  $\psi : \mathcal{O}(X) \rightarrow k_p$ . It is easily seen that a rational subset  $U = R(f_0, \dots, f_n)$  is contained in  $p$  if and only if  $\psi(f_i) \in \psi(f_0)k_p^0$  for all  $1 \leq i \leq n$ . Since  $A = K \cap k_p^0$  it follows that  $p(\mathfrak{p}, A) = p$ .

**Corollary 5.** *Let  $p$  be a prime filter on  $X$  and let  $(\mathfrak{p}, A)$  be the corresponding valuation; we then have:*

- i.  $\mathcal{O}_p$  is a Henselian local ring;
- ii.  $k_p^0$  is Henselian with respect to its prime ideal  $\sqrt{\pi k_p^0}$ ;
- iii.  $\hat{A} = \varprojlim k_p^0 / \pi^n k_p^0$ ; in general  $\hat{A} \neq k_p^0$ ;
- iv. there is a chain of subfields  $\hat{K} \supseteq M \supseteq L$  such that
  - $\hat{K}$  is finite over  $M$ ,
  - $L = k(T_1, \dots, T_d)$  is purely transcendental over  $k$  with  $d \leq \dim(X)$ , and
  - $L$  is  $\pi$ -dense in  $M$ , i.e., for any  $m \in M$  there is a  $\ell \in L$  with  $\ell - m \in \pi \hat{A}$ ;
- v.  $\text{Krull dim}(A) \leq \dim(X) + 1$ .

*Proof.* i. First of all we observe that by Hensel's lemma  $\mathcal{O}_p$  is Henselian if  $(\mathcal{O}_p)_{\text{red}}$  is Henselian. Hence we may assume that  $X$  is reduced. According to one of the equivalent conditions for being Henselian we have to show that

any finite free  $\mathcal{O}_p$ -algebra  $R$  is a product of local rings. In other words we have to show that any idempotent  $\bar{e}$  in  $R \otimes_{\mathcal{O}_p} k_p$  lifts to an idempotent in  $R$ . We clearly find an  $U \in p$  and a finite free  $\mathcal{O}(U)$ -algebra  $R'$  such that  $R = R' \otimes_{\mathcal{O}(U)} \mathcal{O}_p$ . Since  $X$  is reduced  $\|\cdot\|_U$  is a complete norm on  $\mathcal{O}(U)$  ([BGR] 6.2.4.1). According to [BGR] 6.1.1.6 and 6.1.3.3 there is a Banach algebra norm  $\|\cdot\|$  on  $R'$  such that  $(R', \|\cdot\|)$  is a normed  $(\mathcal{O}(U), \|\cdot\|_U)$ -algebra. After replacing  $U$  by a smaller set in  $p$  we may assume that  $\bar{e}$  lifts to an element  $e_0 \in R'$  such that  $\|e_0^2 - e_0\| < 1$ . Define a sequence  $(e_n)_{n \geq 0}$  of elements in  $R'$  by

$$e_{n+1} := 3e_n^2 - 2e_n^3.$$

By construction one has

$$\begin{aligned} e_{n+1} - e_n &= (e_n^2 - e_n)(1 - 2e_n) \quad \text{and} \\ e_{n+1}^2 - e_{n+1} &= 4(e_n^2 - e_n)^3 - 3(e_n^2 - e_n)^2. \end{aligned}$$

It follows that the sequence  $(e_n)_{n \geq 0}$  converges to an idempotent  $e \in R'$  which lifts  $\bar{e}$ .

ii. This is proved in a similar way as the first assertion.

iii. The first statement is easily verified. An example for the inequality is provided by  $X := Sp(k\langle T \rangle)$  and the valuation ring  $A := \left\{ \frac{f}{g} : f, g \in k\langle T \rangle, g \neq 0, \text{ and } \|f\|_X \leq \|g\|_X \right\}$ .

iv. After dividing  $\mathcal{O}(X)$  by a prime ideal we may suppose that  $\phi$  is injective. The field of fractions  $K$  of  $\mathcal{O}(X)$  is  $\pi$ -dense in  $\hat{K}$ . The algebra  $\mathcal{O}(X)$  is finite over some  $R := k\langle T_1, \dots, T_d \rangle$  with  $d$  equal to the dimension of  $X$ . Let  $M \subseteq \hat{K}$  denote the  $\pi$ -completion (in the obvious sense) of the field of fractions of  $R$ . Then  $\hat{K}$  is finite over  $M$  and  $M$  has  $k\langle T_1, \dots, T_d \rangle$  as  $\pi$ -dense subfield.

v. In the valuation ring  $A$  the smallest non-zero prime ideal is  $I := \sqrt{\pi A}$ . Indeed, let  $J \subset I$  be a prime ideal which is smaller than  $I$ . Choose  $a \in J$ . Let  $n \in \mathbb{N}$  and suppose that  $aA + \pi^n A = aA$ . Then  $\pi^n \in aA$  and  $\pi \in J$  and one finds the contradiction  $I \subseteq J$ . Hence  $aA + \pi^n A = \pi^n A$  for all  $n \in \mathbb{N}$ , hence  $a \in \bigcap_{n \in \mathbb{N}} \pi^n A = \{0\}$ , and hence  $J = 0$ .

We will show that  $\text{Krull dim}(A/I) \leq \dim(X)$ . The map  $\phi : \mathcal{O}(X) \rightarrow K$  has the properties  $\phi(\mathcal{O}(X)^0) \subseteq A$  and  $\phi(\mathcal{O}(X)^{00}) \subseteq I$ . The kernel of the induced homomorphism  $\mathcal{O}(\bar{X}) \rightarrow A_1 := A/I$  is a prime ideal corresponding to a closed irreducible (and reduced) subset  $Y \subseteq \bar{X}$ . It can be seen that the induced injective map  $\mathcal{O}(Y) \rightarrow A_1$  gives a bijection between the fields of fractions of the two rings. In other terms:  $A_1$  is a valuation ring of the field of fractions of  $\mathcal{O}(Y)$  containing  $\mathcal{O}(Y)$ . It is well known that the Krull dimension of such a valuation ring is  $\leq$  the dimension of  $Y$ .

An ordinary point  $x \in X$  can be identified with the prime filter  $\{U : x \in U\}$ . The valuation corresponding to  $x$  is given by the homomorphism  $\mathcal{O}(X) \rightarrow \mathcal{O}(X)/\mathfrak{m}_x$  where  $\mathfrak{m}_x$  denotes the maximal ideal corresponding to  $x$ . In this way  $X$  can be embedded in  $\mathcal{P}(X)$  and  $\text{Val}(X)$ , respectively.

A valuation  $(p, A)$  on  $X$  is called an *analytic point* of  $X$  if the valuation ring  $A$  has rank 1. Recall that a valuation ring has rank 1 if and only if its



Krull dimension is 1 ([M] 10.7). Obviously any ordinary point gives rise to an analytic point. Let  $p_1$  and  $p_2$  be two prime filters on  $X$  with corresponding valuations  $(p_i, A_i)$ . Then  $p_1 \subseteq p_2$  if and only if  $p_1 = p_2$  and  $A_2$  is a localization of  $A_1$  with respect to some prime ideal of  $A_1$ . This follows easily from Theorem 4. In particular the prime filter  $p$  is maximal if and only if the corresponding valuation ring  $A$  has rank 1. In other words: The subset of analytic points corresponds to the subset  $\mathcal{M}(X)$  of maximal filters.

The analytic points of  $X$  can also be described in the following way. An analytic point  $a$  of  $X$  is a semi-norm  $|\cdot|_a : \mathcal{O}(X) \rightarrow \mathbb{R}_{\geq 0}$  on the affinoid algebra  $\mathcal{O}(X)$  of  $X$  satisfying:

- (a1)  $|f + g|_a \leq \max(|f|_a, |g|_a)$  for all  $f, g \in \mathcal{O}(X)$ ;
- (a2)  $|fg|_a = |f|_a |g|_a$  for all  $f, g \in \mathcal{O}(X)$ ;
- (a3)  $|\lambda|_a = |\lambda|$  for any  $\lambda \in k$ ;
- (a4)  $|\cdot|_a : \mathcal{O}(X) \rightarrow \mathbb{R}_{\geq 0}$  is continuous with respect to the norm topology on  $\mathcal{O}(X)$ .

Still another characterization of analytic points will be given in Lemma 6.

As an example we give an explicit description of the prime filters on the disk  $D = Sp(k\langle T \rangle)$ . For convenience we suppose that  $k$  is algebraically closed. For any  $d \in D$  and any  $\rho \in |k^\times|$  with  $0 < \rho \leq 1$  we write  $D(d, \rho) := \{x \in D : |x - d| \leq \rho\}$ . This is a closed disk. We will write  $D(d, \rho)^*$  for any subset of  $D$  of the form

$$\{x \in D(d, \rho) : |x - a_i| = \rho \text{ for } i = 1, \dots, m\} \quad \text{with } a_1, \dots, a_m \in D(d, \rho).$$

This is a formal open subset of  $D(d, \rho)$ . We start by describing the analytic points of  $D$  (compare also [B] 1.4.4).

Let  $|\cdot|_a$  be an analytic point. For every  $d \in D$  one defines the real number  $\rho(d) := |T - d|_a$ . Then  $D(d, \rho)$  belongs to  $a$  if and only if  $\rho(d) \leq \rho$ . This follows from the definition of the filter attached to  $|\cdot|_a$ .

For convenience we extend the notation, namely  $D(d, \rho) := \{x \in D : |x - d| \leq \rho\}$  for any  $\rho \in \mathbb{R}$ ,  $0 \leq \rho \leq 1$ .

For two points  $d_1, d_2 \in D$  one has

$$D(d_1, \rho(d_1)) \subseteq D(d_2, \rho(d_2)) \text{ or } D(d_2, \rho(d_2)) \subseteq D(d_1, \rho(d_1))$$

because  $|d_1 - d_2| = |(T - d_1) - (T - d_2)|_a \leq \max(|T - d_1|_a, |T - d_2|_a)$ . Now there are several possibilities: If  $\bigcap \{D(d, \rho(d)) : d \in D\}$  is

(1) an (ordinary) point  $x \in D$  then  $a$  is the filter of all special subsets containing  $x$ ;

(2) equal to  $D(d_0, \rho_0)$  with  $\rho_0 \in |k^\times|$  then  $a$  consists of the special subsets containing some  $D(d_0, \rho_0)^*$ ; further  $|\cdot|_a$  is the spectral norm (or supremum norm) on  $D(d_0, \rho_0)$ ;

(3) equal to  $D(d_0, \rho_0)$  with  $\rho_0 \notin |k^\times|$  then  $|\cdot|_a$  still is the supremum norm on  $D(d_0, \rho_0)$ ; further  $D(d, \rho)^*$  belongs to  $a$  if and only if  $D(d, \rho)^* \supseteq \{x \in D : \rho_1 \leq |x - d_0| \leq \rho_2\}$  with  $\rho_1 < \rho_0 < \rho_2$  and  $\rho_1, \rho_2 \in |k^\times|$ ;

(4) empty then  $|\cdot|_a$  is the infimum of the supremum norms taken over the sets  $D(d, \rho(d))$ ; a special subset belongs to  $a$  if and only if it contains some  $D(d, \rho(d))$ ; this case can only occur if the field  $k$  is not maximally complete.

The description of the prime filters  $p$  which are not maximal is more complicated. We will use the results and notations of the subsequent paragraph for this. Let  $a := r(p)$ . For an  $a$  of the type (1), (3), or (4) above one sees that  $\tilde{a} = a$ ; hence  $p = a$ . For  $a$  of type (2), we take for notational convenience  $D = (d_0, \rho_0)$ . For any  $\bar{\lambda} \in \bar{k}$  one chooses a  $\lambda \in k^0$  with residue  $\bar{\lambda}$ . Define  $p_{\bar{\lambda}}$  to be the family of special subsets of  $D$  containing for some  $\rho \in |k^\times|$ ,  $\rho < 1$ , the ring domain  $\{x \in D : \rho < |x - \lambda| < 1\}$ . This is in fact a prime filter and  $r(p_{\bar{\lambda}}) = a$ . Using Lemma 6 one can show that any  $p \neq a$  with  $r(p) = a$  is equal to  $p_{\bar{\lambda}}$  for a unique  $\bar{\lambda} \in \bar{k}$ . This finishes the description of all prime filters on  $D$ .

Coming back to the general situation we finally describe a natural retraction map

$$r : \mathcal{P}(X) \rightarrow \mathcal{M}(X).$$

Let  $p$  be any prime filter on  $X$ . The map  $f \mapsto \|f\|_p$  satisfies (a1)–(a4), as is easily seen. We denote by  $r(p)$  the analytic point of  $X$  (resp. the maximal filter) which corresponds to  $\|\cdot\|_p$ . One has that  $R(f_1, \dots, f_n)$  belongs to  $r(p)$  if and only if  $\|f_0\|_p \geq \|f_i\|_p$  for all  $1 \leq i \leq n$ . Moreover, this is equivalent to  $R(\rho f_0, f_1, \dots, f_n) \in p$  for every  $\rho \in \sqrt{|k^\times|}$ ,  $\rho > 1$ . The notation which we use here and in the sequel is

$$R(\rho f_0, f_1, \dots, f_n) := \{x \in X : \rho |f_0(x)| \geq |f_i(x)| \text{ for all } 1 \leq i \leq n\};$$

this is a rational subset ([BGR] 7.2.3).

**Lemma 6.** i. A prime filter  $p$  is maximal if and only if it has the property that  $R(f_0, \dots, f_n) \in p$  if  $R(\rho f_0, f_1, \dots, f_n) \in p$  for every  $\rho \in \sqrt{|k^\times|}$ ,  $\rho > 1$ ;

ii. for a prime filter  $p$  the filter  $r(p)$  is the unique maximal filter containing  $p$ ;

iii. if  $A$  is the valuation ring corresponding to the prime filter  $p$  then the valuation ring corresponding to  $r(p)$  is the localization of  $A$  at the prime ideal  $\sqrt{\pi}A$ .

*Proof.* Obvious.

We will need some further properties of this retraction map.

**Lemma 7.** Let  $a$  be an analytic point of  $X$  with corresponding valuation  $(\mathfrak{p}, A)$ ; we have:

i. define  $\tilde{a}$  to be the family of all special subsets of  $X$  which contain  $R(\rho f_0, f_1, \dots, f_n)$  for some  $R(f_0, \dots, f_n) \in a$  and some  $\rho \in \sqrt{|k^\times|}$ ,  $\rho > 1$ ; then  $\tilde{a}$  is equal to the intersection of all prime filters  $p$  with  $r(p) = a$ ;

ii. the set  $r^{-1}(\{a\})$  can be identified with the set of all valuation rings in the residue field  $A/\mathfrak{m}_A$  of  $A$  which contain the image of the map  $\mathcal{O}(X)^0 \rightarrow \mathcal{O}(\bar{X}) \rightarrow A/\mathfrak{m}_A$ .

*Proof.* i. We know already that the prime filters  $p$  containing  $\tilde{a}$  are precisely those with  $r(p) = a$ . In particular  $\tilde{a}$  is contained in the intersection of all those

prime filters. In order to see the reverse inclusion observe first that  $\tilde{a}$  is a filter. If  $U \notin \tilde{a}$  is any special subset then applying Remark 1 to the family  $s := \{U\}$  we obtain a prime filter containing  $\tilde{a}$  but not  $U$ . ii. This follows from Lemma 6.iii and [M] 10.1.

### 3 Topologies on $\mathcal{P}(X)$ and $\mathcal{M}(X)$

Let  $Z$  be a scheme of finite type over a field  $F$ . We denote by  $Z_{cl}$  the variety over  $F$  consisting of the closed points of  $Z$ . A prime filter on  $Z_{cl}$  is a collection of Zariski open subsets of  $Z_{cl}$  having the properties (p1)–(p4) as in Sect. 2. Let  $\mathcal{P}(Z_{cl})$  be the set of all those prime filters. Every point  $z \in Z$  defines an irreducible closed subset  $\overline{\{z\}}$  and a prime filter  $p_z$  consisting of the open subsets  $U \subseteq Z_{cl}$  such that  $U \cap \{z\} \neq \emptyset$ . We want to prove the following amusing fact.

**Lemma 1.** *The map  $Z \rightarrow \mathcal{P}(Z_{cl})$  given by  $z \mapsto p_z$  is a bijection.*

*Proof.* The injectivity of the map is clear. For the surjectivity it suffices to consider an affine scheme  $Z$ . Let  $p$  be a prime filter on  $Z_{cl}$  and define

$$\mathcal{O}_p := \varinjlim_{U \in p} \mathcal{O}(U) \quad \text{and} \quad \mathfrak{m}_p := \varinjlim_{U \in p} \{f \in \mathcal{O}(U) : \{y \in U : f(y) \neq 0\} \notin p\}.$$

Here  $\mathcal{O}$  denotes the structure sheaf of  $Z$  as well as its inverse image on  $Z_{cl}$ . Clearly  $\mathfrak{m}_p$  is an ideal in  $\mathcal{O}_p$ . If  $f \in \mathcal{O}_p \setminus \mathfrak{m}_p$  is defined on  $U \in p$  then the set  $\{y \in U : f(y) \neq 0\}$  belongs to  $p$  and so  $f$  is invertible in  $\mathcal{O}_p$ . Hence  $\mathcal{O}_p$  is a local ring. The kernel of  $\phi : \mathcal{O}(Z) \rightarrow \mathcal{O}_p \rightarrow \mathcal{O}_p/\mathfrak{m}_p$  is some prime ideal  $I$  of  $\mathcal{O}(Z)$ . For  $f \in \mathcal{O}(Z)$  put  $Z_f := \{y \in Z : f(y) \neq 0\}$  which is Zariski open in  $Z$ ; then  $Z_f \cap Z_{cl} \in p$  if and only if  $f \notin I$ . Since the  $Z_f$  form a basis of the Zariski topology it follows from (p4) that  $\mathcal{O}_p$  is the direct limit of all  $\mathcal{O}(Z_f)$  with  $f \notin I$ . Therefore  $\mathcal{O}_p$  is the localization of  $\mathcal{O}(Z)$  with respect to the prime ideal  $I$ . The prime ideal  $I$  is a point  $z$  of  $Z$  and  $p = p_z$ .

On  $\mathcal{P}(Z_{cl})$  we put the topology induced by its bijection with  $Z$ . One can verify that the open sets in  $\mathcal{P}(Z_{cl})$  are the sets  $\{p \in \mathcal{P}(Z_{cl}) : U \in p\}$  where  $U \subseteq Z_{cl}$  is a Zariski open set.

We return now to the affinoid space  $X$  over  $k$  and its set of prime filters  $\mathcal{P}(X)$ . On this set we define a topology by taking as a basis for the open sets the sets  $\tilde{U} := \{p \in \mathcal{P}(X) : U \in p\}$  where  $U \subseteq X$  is a special subset. If  $U$  is an affinoid subdomain of  $X$  then the obvious bijection  $\mathcal{P}(U) \xrightarrow{\sim} \tilde{U}$  is a homeomorphism. The next fact is a central result in [H].

**Lemma 2.** i. *The space  $\mathcal{P}(X)$  is quasi-compact;*  
ii. *for any analytic point  $a \in \mathcal{M}(X) \subseteq \mathcal{P}(X)$  the closure of  $\{a\}$  is the set  $r^{-1}(\{a\})$ , where  $r$  is the retraction map.*

*Proof.* i. Let  $\{\tilde{U}_j\}_{j \in J}$  be a covering of  $\mathcal{P}(X)$ . Assume that this covering has no finite subcovering. By (p4) this means that  $X$  does not belong to the family

$s$  of all finite unions of sets  $U_j$ . Applying Remark 2.1 to the filter  $\{X\}$  we obtain a prime filter  $p$  such that  $p \cap s = \emptyset$ . On the other hand we have  $p \in \tilde{U}_{j_0}$  for some  $j_0 \in J$  and hence  $U_{j_0} \in p \cap s$ . This is a contradiction. ii. Let a prime filter  $p$  be given. By the definition of the topology we have

$$\mathcal{P}(X) \setminus \overline{\{p\}} = \{p' : p' \text{ contains some special subset } U \notin p\}$$

and hence  $\overline{\{p\}} = \{p' : p' \subseteq p\}$ . The assertion follows now from Lemma 2.6.ii.

The maximal ideal corresponding to a point in  $X$  is the kernel of a surjective  $k$ -algebra homomorphism  $\psi : \mathcal{O}(X) \rightarrow \ell$  where  $\ell$  is a finite extension of  $k$ . There is an induced  $\bar{k}$ -algebra homomorphism  $\bar{\psi} : \mathcal{O}(\bar{X}) \rightarrow \bar{\ell}$ . The kernel of  $\bar{\psi}$  is a maximal ideal of  $\mathcal{O}(\bar{X})$ . This defines a map

$$\text{red} : X \rightarrow \bar{X}_{cl}$$

which is called the *canonical reduction map*. It is surjective and for every Zariski open  $V \subseteq \bar{X}_{cl}$  the preimage  $\text{red}^{-1}V$  is a special subset in  $X$  ([BGR] 7.1.5.2 and 4).

**Lemma 3.** *There is a natural surjective continuous map  $\text{red} : \mathcal{P}(X) \rightarrow \mathcal{P}(\bar{X}_{cl}) \cong \bar{X}$ .*

*Proof.* For  $p \in \mathcal{P}(X)$  define

$$\text{red}(p) := \{V \subseteq \bar{X}_{cl} \text{ open} : \text{red}^{-1}V \in p\}.$$

It is clear that  $\text{red}(p)$  is a prime filter on  $\bar{X}_{cl}$ . The continuity of  $\text{red}$  follows from the definition of the topologies. Let a prime filter  $q$  on  $\bar{X}_{cl}$  be given. The family

$$f := \text{all special subsets } U \subseteq X \text{ such that } U \supseteq \text{red}^{-1}V \text{ for some } V \in q$$

is a filter on  $X$ . The family

$$s := \text{all special subsets } U \subseteq X \text{ such that } U \subseteq \text{red}^{-1}W$$

$$\text{for some open } W \subseteq \bar{X}_{cl} \text{ with } W \notin q$$

is closed with respect to finite unions and fulfills  $f \cap s = \emptyset$ . Applying the Remark 2.1 we obtain a prime filter  $p$  on  $X$  such that  $p \supseteq f$  and  $p \cap s = \emptyset$ . The former property implies that  $q \subseteq \text{red}(p)$  and the latter that  $\text{red}(p) \subseteq q$ .

These considerations can be generalized in the following way. Consider the rational covering  $\{U_0, \dots, U_n\}$  of  $X$  given by elements  $f_0, \dots, f_n \in \mathcal{O}(X)$  generating the unit ideal. For every  $i, j$  one has that  $\bar{U}_i \cap \bar{U}_j$  is an open affine subscheme of  $\bar{U}_i$  ([BGR] 7.2.6.3). The affine schemes  $\bar{U}_i$  are glued together over these open subschemes. The result is a reduced scheme of finite type over  $\bar{k}$  which we denote by  $(\bar{X}, f.)$ . The canonical reduction maps  $U_i \rightarrow (\bar{U}_i)_{cl}$  glue to a map

$$\text{red}(f.) : X \rightarrow \overline{(\bar{X}, f.)}_{cl}$$

which is called the *reduction map of the rational covering* given by  $f. = \{f_0, \dots, f_n\}$ . Again it is surjective and for every open  $V \subseteq \overline{(\bar{X}, f.)}_{cl}$  the

preimage  $\text{red}(f.)^{-1}V$  is a special subset in  $X$ . As in Lemma 3 one can prove that this map extends naturally to a surjective continuous map

$$\text{red}(f.) : \mathcal{P}(X) \rightarrow \mathcal{P}(\overline{(X, f.)}_{cl}) \cong \overline{(X, f.)}.$$

This leads to a continuous map

$$\mathcal{P}(X) \rightarrow \varprojlim \overline{(X, f.)}$$

in which the last space is provided with the topology of the projective limit.

**Theorem 4.**  $\mathcal{P}(X) \xrightarrow{\sim} \varprojlim \overline{(X, f.)}$  is a (bijective) homeomorphism.

*Proof.* First note that for any rational set  $U = R(g_0, \dots, g_m)$  in  $X$  the subset  $\bar{U}_{cl}$  in  $\overline{(X, g.)}_{cl}$  is open and  $U = \text{red}(g.)^{-1}\bar{U}_{cl}$ . This immediately implies that the map in question is injective. But it also shows that for any  $\{p(f.)\}_f$  in the projective limit on the right hand side the filter

$$p := \text{all special subsets } U \subseteq X \text{ such that } U \supseteq \text{red}(f.)^{-1}V \\ \text{for some } f. \text{ and some } V \in p(f.)$$

on  $X$  is a prime filter. One easily checks that  $p$  is a preimage of  $\{p(f.)\}_f$ . Hence our map is bijective. In order to see that the map is open let  $U = R(g_0, \dots, g_m)$  be again a rational set in  $X$ . The image of  $\bar{U} \subseteq \mathcal{P}(X)$  consists of the elements  $\{p(f.)\}_f$  in the projective limit such that  $p(g.)$  contains the open set  $\bar{U}_{cl} \subseteq \overline{(X, g.)}_{cl}$ .

Assume the valuation of  $k$  to be discrete and let  $\pi$  generate the maximal ideal of  $k^0$ . The Raynaud functor associates to every formal scheme  $\mathcal{X}$  of finite type and flat over  $k^0$  a rigid analytic space  $X := \mathcal{X} \otimes k$  called the “generic fiber” of  $\mathcal{X}$ . If the formal scheme is affine, i.e.,  $\mathcal{X}$  is the formal spectrum of a flat  $k^0$ -algebra of the type  $R = k^0\langle T_1, \dots, T_n \rangle / I$  then  $X$  is the affinoid space with algebra  $\mathcal{O}(X) = R \otimes_{k^0} k$ . One finds a reduction map  $\text{red} : X \rightarrow \bar{\mathcal{X}}_{cl}$  where  $\bar{\mathcal{X}} := \text{Spec}(R \otimes_{k^0} \bar{k})$ . In the general case one has a similar reduction map where  $\bar{\mathcal{X}} := \mathcal{X} \otimes_{k^0} \bar{k}$  is the special fiber of  $\mathcal{X}$ . Blowing ups of  $\mathcal{X}$  with respect to an ideal with support in the special fiber do not change  $X$  but change the reduction map.

Let  $X$  be a reduced affinoid space over  $k$ . The ring  $\mathcal{O}(X)^0$  is noetherian,  $\pi$ -adically complete and  $\mathcal{O}(X)^0/(\pi)$  is of finite type over  $\bar{k}$  ([BGR] 6.4.1.6). This makes  $\mathcal{X} := \text{Spf}(\mathcal{O}(X)^0)$  into a formal scheme of finite type and flat over  $k^0$  with generic fiber  $X$ . The reduction map corresponding to  $\mathcal{X}$  is what we have called the canonical reduction map. For any rational covering  $\{U_1, \dots, U_n\}$  of  $X$  given by elements  $f_0, \dots, f_n$  the glueing of the  $\text{Spf}(\mathcal{O}(U_i)^0)$  gives a formal scheme  $\mathcal{X}(f.)$  of finite type and flat over  $k^0$  with generic fiber  $X$  and with reduction as explained before Theorem 4. This new formal scheme can be obtained from  $\mathcal{X}$  by blowing up some ideal supported in the special fiber. One can show that the family  $\mathcal{X}(f.)$  is cofinal in the collection of all formal

schemes  $\mathcal{U}$  of the type above with  $X$  as generic fiber. This is the interpretation of Theorem 4 in terms of formal schemes.

For more general fields this interpretation remains valid but the formulation is more complicated due to the fact that the rings  $k^0\langle T_1, \dots, T_n \rangle$  are not noetherian.

**Definition.** *The Berkovich topology on  $\mathcal{M}(X)$  is the coarsest topology such that, for every  $f \in \mathcal{O}(X)$ , the function  $a \mapsto |f|_a$  on  $\mathcal{M}(X)$  is continuous.*

The description of analytic points in terms of semi-norms on  $\mathcal{O}(X)$  shows that the functions in the above Definition separate points in  $\mathcal{M}(X)$ . Hence the Berkovich topology is Hausdorff. In the following  $\mathcal{M}(X)$  always is equipped with the Berkovich topology.

**Theorem 5.** *The Berkovich topology on  $\mathcal{M}(X)$  coincides with the quotient topology derived from the topology on  $\mathcal{P}(X)$  and the retraction map  $r : \mathcal{P}(X) \rightarrow \mathcal{M}(X)$ ;  $\mathcal{M}(X)$  is compact.*

*Proof.* We temporarily write  $\mathcal{M}(X)_B$  for  $\mathcal{M}(X)$  equipped with the Berkovich topology. It suffices to show that the map  $r : \mathcal{P}(X) \rightarrow \mathcal{M}(X)_B$  is continuous. Since the first space is quasi-compact and hence the second one is compact the map then must be a quotient map. It is easy to see that the open subsets

$$M(f_0, \dots, f_n) := \{a \in \mathcal{M}(X) : |f_0|_a > |f_i|_a \text{ for all } 1 \leq i \leq n\}$$

for any  $f_0, \dots, f_n \in \mathcal{O}(X)$  generating the unit ideal form a basis of the topology of  $\mathcal{M}(X)_B$ . Let  $p$  be prime filter on  $X$ . We have  $r(p) \in M(f_0, \dots, f_n)$  if and only if  $R(\rho f_0, f_1, \dots, f_n) \in r(p)$  for some  $\rho \in \sqrt{|k^\times|}$ ,  $\rho < 1$ , which by Lemma 2.7.i is equivalent to  $R(\rho' f_0, f_1, \dots, f_n) \in p$  for all  $\rho' > \rho$ . It follows that

$$(*) \quad r^{-1}M(f_0, \dots, f_n) = \bigcup \{R(\rho f_0, f_1, \dots, f_n)^\sim : \rho \in \sqrt{|k^\times|} \text{ and } \rho < 1\}.$$

The right hand side is open in  $\mathcal{P}(X)$ .

The second statement in the above Theorem is due to Berkovich ([B]). But our proof is completely different.

#### 4 Sheaves on $X$ , $\mathcal{M}(X)$ , and $\mathcal{P}(X)$

The inclusion  $X \subseteq \mathcal{P}(X)$  is not very useful for comparing sheaves on the two spaces. But viewing  $X$  with its Grothendieck topology as well as the topological space  $\mathcal{P}(X)$  as sites the functor  $U \mapsto \tilde{U}$  defines a morphism of sites

$$\sigma : \mathcal{P}(X) \rightarrow X.$$

The following result was proved in [H].

**Theorem 1.** *The functors  $\sigma_*$  and  $\sigma^*$  are quasi-inverse equivalences between the categories of abelian sheaves on  $\mathcal{P}(X)$  and on  $X$ .*

*Proof.* Let  $F$  be an abelian sheaf on  $X$ . Then  $\sigma^*F$  is the sheaf associated with the presheaf  $P$  given by

$$P(N) = \varinjlim_{N \subseteq \tilde{U}} F(U) \text{ for any open } N \subseteq \mathcal{P}(X).$$

Consider the case where  $N$  is of the form  $N = \tilde{V}$  for some special subset  $V \subseteq X$ . A special subset  $U$  such that  $\tilde{V} \subseteq \tilde{U}$  satisfies  $V \subseteq U$ . Indeed, suppose there is a point  $x \in V \setminus U$ . Its neighbourhood filter lies in  $\tilde{V}$  but does not contain  $U$  which is a contradiction. It follows that  $P(\tilde{V}) = F(V)$ . In particular  $P$  already satisfies the sheaf axiom for coverings consisting of open sets of the form  $\tilde{V}$ . Since the latter form a basis of the topology of  $\mathcal{P}(X)$  sheafification leaves the sections in such a set  $\tilde{V}$  unchanged: We have

$$(\sigma_*\sigma^*F)(V) = (\sigma^*F)(\tilde{V}) = P(\tilde{V}) = F(V) \text{ and hence } \sigma_*\sigma^*F = F.$$

Also for an abelian sheaf  $S$  on  $\mathcal{P}(X)$  we obtain

$$(\sigma^*\sigma_*S)(\tilde{V}) = (\sigma_*S)(V) = S(\tilde{V}) \text{ and hence } \sigma^*\sigma_*S = S.$$

This proof in particular shows that for any abelian sheaf  $F$  on  $X$  and any special subset  $U \subseteq X$  we have

$$(\sigma^*F)(\tilde{U}) = F(U).$$

It follows that

$$F_p := \varinjlim_{U \in p} F(U) = (\sigma^*F)_p \text{ for any } p \in \mathcal{P}(X)$$

where the right hand side is the stalk in the usual sense of the sheaf  $\sigma^*F$  in the point  $p$ . As a consequence we obtain that the functors  $F \mapsto F_p$  are exact and that  $F = 0$  if all  $F_p = 0$ .

An abelian (pre)sheaf  $F$  on  $X$  is called *overconvergent* if, for all  $f_0, \dots, f_n \in \mathcal{O}(X)$  generating the unit ideal, we have

$$F(R(f_0, \dots, f_n)) = \varinjlim F(R(\rho f_0, f_1, \dots, f_n))$$

where the limit is taken over all  $\rho \in \sqrt{|k^\times|}$ ,  $\rho > 1$ . One can verify that this is the same notion as that of a conservative sheaf in [S] as well as that of a constructible sheaf in [P]. We note that the sheaf associated to an overconvergent presheaf is also overconvergent. A sheaf on  $\mathcal{P}(X)$  will be called *overconvergent* if the corresponding sheaf on  $X$  is overconvergent. The following theorem is one of the main results in [S]. The proof given here is however more direct.

**Theorem 2.** *The retraction map  $r : \mathcal{P}(X) \rightarrow \mathcal{M}(X)$  gives rise to quasi-inverse equivalences  $r_*$  and  $r^*$  between the category of overconvergent sheaves on  $\mathcal{P}(X)$  (or on  $X$ ) and the category of all abelian sheaves on  $\mathcal{M}(X)$ .*

*Proof.* (1) First we show that for any abelian sheaf  $T$  on  $\mathcal{M}(X)$  the sheaf  $r^*T$  is overconvergent. Since sheafification preserves the property of being overconvergent it suffices to check that the presheaf inverse image of  $T$  is overconvergent. This is the presheaf  $P$  on  $\mathcal{P}(X)$  given by

$$P(N) = \varinjlim_{N \subseteq r^{-1}M} T(M).$$

Let  $U = R(f_0, \dots, f_n)$  be a rational subset of  $X$ . Then

$$r(\tilde{U}) = \{a \in \mathcal{M}(X) : U \in a\} = \{a \in \mathcal{M}(X) : |f_0|_a \geq |f_1|_a, \dots, |f_n|_a\}$$

is a closed subset in  $\mathcal{M}(X)$ . A fundamental system of open neighbourhoods of  $r(\tilde{U})$  is given by  $\{M(\varepsilon) : \varepsilon > 0\}$  with

$$M(\varepsilon) := \{a \in \mathcal{M}(X) : (1 + \varepsilon)|f_0|_a > |f_1|_a, \dots, |f_n|_a\}.$$

We have  $P(\tilde{U}) = \varinjlim T(M(\varepsilon))$  and this implies that

$$P(\tilde{U}) = \varinjlim P(R(\rho f_0, f_1, \dots, f_n)^\sim)$$

where the direct limit is taken over all  $\rho \in \sqrt{|k^\times|}$ ,  $\rho > 1$ . The presheaf  $P$  therefore is overconvergent.

(2) Next we show that for any overconvergent sheaf  $S$  on  $\mathcal{P}(X)$  and any analytic point  $a$  of  $X$  the natural map

$$(r_*S)_a \xrightarrow{\cong} S_a$$

is bijective. By the construction of the topology  $S_a$  is the direct limit of all  $S(\tilde{U})$  with  $U = R(f_0, \dots, f_n) \in a$ . Since  $S$  is overconvergent  $S_a$  is also the direct limit of all  $S(\tilde{U})$  where  $U = R(\rho f_0, f_1, \dots, f_n)$  with  $R(f_0, \dots, f_n) \in a$  and  $\rho \in \sqrt{|k^\times|}$ ,  $\rho > 1$ . Our claim follows since the open subset

$$M := \{b \in \mathcal{M}(X) : \rho|f_0|_b > |f_1|_b, \dots, |f_n|_b\}$$

in  $\mathcal{M}(X)$  satisfies  $r^{-1}M \subseteq R(\rho f_0, f_1, \dots, f_n)^\sim$  by the formula  $(*)$  in the proof of Theorem 3.5.

(3) A homomorphism of overconvergent sheaves  $S \rightarrow S'$  is an isomorphism if for any analytic point  $a \in \mathcal{P}(X)$  the homomorphism  $S_a \rightarrow S'_a$  is bijective. Indeed, let  $p$  be any prime filter contained in the maximal filter  $a$ . Then the natural map  $S_p \rightarrow S'_p$  is an isomorphism for any overconvergent sheaf  $S$ .

(4) For any abelian sheaf  $T$  on  $\mathcal{M}(X)$  we have  $(r_*r^*T)_a = (r^*T)_a$  by (1) and (2) and  $(r^*T)_a = T_a$  since  $r(a) = a$ . Hence  $r_*r^*T = T$ .

For any overconvergent sheaf  $S$  on  $\mathcal{P}(X)$  and any analytic point  $a \in \mathcal{P}(X)$  we have  $(r^*r_*S)_a = (r_*S)_a = S_a$  according to (2). Because of (1) and (3) this proves that  $r^*r_*S = S$ .

**Corollary 3.** *Let  $S$  be a sheaf on  $\mathcal{P}(X)$  and  $a$  an analytic point of  $X$ ; we have*



$$(r^*r_*S)_a = (r_*S)_a = S_{\tilde{a}} := \varinjlim_{U \in \tilde{a}} S(\tilde{U}) .$$

*Proof.* ( $\tilde{a}$  was defined in Lemma 2.7.) Consider the two families

$$\{\tilde{U} : U \in \tilde{a}\}$$

and

$$\{r^{-1}M(f_0, \dots, f_n) : f_0, \dots, f_n \in \mathcal{O}(X) \text{ generating the unit ideal} \\ \text{such that } a \in M(f_0, \dots, f_n)\}$$

of open subsets in  $\mathcal{P}(X)$ . It is an immediate consequence of the formula (\*) in the proof of Theorem 3.5 that every member of the second family contains a member of the first family. The same formula (\*) also shows, as noted already in the previous proof, the following: If  $R(f_0, \dots, f_n) \in a$  so that  $R(\rho f_0, f_1, \dots, f_n) \in \tilde{a}$  for any  $\rho \in \sqrt{|k^\times|}$ ,  $\rho > 1$ , then

$$r^{-1}\{b \in \mathcal{M}(X) : \rho|f_0|_b > |f_i|_b \text{ for all } 1 \leq i \leq n\} \subseteq R(\rho f_0, f_1, \dots, f_n)^\sim .$$

Hence any set in the first family contains a set in the second family.

## 5 General rigid spaces

In this section  $X$  is an arbitrary rigid space over  $k$ . The definition of the sets  $\mathcal{M}(X) \subseteq \mathcal{P}(X)$  and  $\text{Val}(X)$  together with the natural bijection between  $\text{Val}(X)$  and  $\mathcal{P}(X)$  generalizes in a straightforward way. Of course filters on  $X$  now have to be formed among all admissible open subsets of  $X$  and prime filters have to be defined by the condition (p4)'. For any affinoid open subset  $U \subseteq X$  there are obvious bijections

$$\mathcal{P}(U) \xrightarrow{\sim} \tilde{U} := \{p \in \mathcal{P}(X) : U \in p\}$$

and

$$\mathcal{M}(U) \xrightarrow{\sim} \underline{U} := \{a \in \mathcal{M}(X) : U \in a\} .$$

The retraction map

$$r_X : \mathcal{P}(X) \rightarrow \mathcal{M}(X) \\ p \mapsto \begin{array}{l} \text{unique maximal filter} \\ \text{containing } p \end{array}$$

still is defined. We equip  $\mathcal{P}(X)$  with the topology for which the subsets  $\tilde{U}$  for  $U \subseteq X$  affinoid open form a base. Then  $\mathcal{P}(U) \xrightarrow{\sim} \tilde{U}$  is a homeomorphism. We have

$$r_X^{-1}(\{a\}) = \text{closure of } \{a\} \text{ in } \mathcal{P}(X) \quad \text{for any } a \in \mathcal{M}(X) .$$

It remains true in general that the categories of abelian sheaves on  $\mathcal{P}(X)$  and on  $X$  are naturally equivalent. We always give  $\mathcal{M}(X)$  the quotient topology with respect to the map  $r_X$ .

**Lemma 1.** *Assume  $X$  to be affinoid; then the subsets  $\mathcal{M}(X) \setminus \underline{U}$  with  $U$  running through the affinoid subdomains of  $X$  are open and generate the topology of  $\mathcal{M}(X)$ .*

*Proof.* Clearly the natural map  $\mathcal{M}(U) \rightarrow \mathcal{M}(X)$  is continuous. Since both sides are compact its image  $\underline{U}$  is closed. Using the Berkovich topology we know that the sets  $\{a \in \mathcal{M}(X) : |f|_a < \rho\}$  and  $\{a \in \mathcal{M}(X) : |f|_a > \rho\}$  with  $f \in \mathcal{O}(X)$  and  $\rho \in \sqrt{|k^\times|}$  generate the topology of  $\mathcal{M}(X)$ . Those sets obviously are of the form  $\mathcal{M}(X) \setminus \underline{U}$ .

**Lemma 2.** *Assume  $X$  to be quasi-separated and let  $U \subseteq X$  be affinoid open; then the natural map  $\mathcal{M}(U) \rightarrow \mathcal{M}(X)$  is a homeomorphism onto its image  $\underline{U}$  which is closed in  $\mathcal{M}(X)$ .*

*Proof.* Let  $X = \bigcup_{i \in I} U_i$  be an admissible affinoid open covering. Making obvious identifications we have

$$\begin{aligned} \mathcal{P}(X) \setminus r_X^{-1} \underline{U} &= \bigcup_{i \in I} \mathcal{P}(U_i) \setminus r_X^{-1} \underline{U} = \bigcup_{i \in I} \mathcal{P}(U_i) \setminus r_{U_i}^{-1} (\underline{U} \cap \underline{U}_i) \\ &= \bigcup_{i \in I} r_{U_i}^{-1} (\mathcal{M}(U_i) \setminus (\underline{U} \cap \underline{U}_i)). \end{aligned}$$

Since  $X$  is quasi-separated it follows from Lemma 1 that  $\mathcal{M}(U_i) \setminus (\underline{U} \cap \underline{U}_i)$  is open in  $\mathcal{M}(U_i)$ . This implies that  $\underline{U}$  is closed in  $\mathcal{M}(X)$ . In this way we see that the natural continuous map  $\mathcal{M}(U) \rightarrow \mathcal{M}(X)$  has the property that the image  $\underline{Y}$  of  $\mathcal{M}(V)$  for any affinoid subdomain  $V \subseteq U$  is closed in  $\mathcal{M}(X)$ . Again by Lemma 1 this map therefore is closed.

**Lemma 3.** *Let  $X = \bigcup_{i \in I} U_i$  be an admissible affinoid open covering; we then have: A subset  $M$  of  $\mathcal{M}(X)$  is open (closed) if and only if the preimage of  $M$  in  $\mathcal{M}(U_i)$  is open (closed) for any  $i \in I$ .*

*Proof.* The direct implication is trivial. Also the assertion about closedness follows from the one about openness. Therefore assume that the preimage of  $M$  in  $\mathcal{M}(U_i)$  is open for any  $i \in I$ . We have to show that  $r_X^{-1} M$  is open. But (again with the obvious identifications)

$$r_X^{-1} M = \bigcup_{i \in I} (r_X^{-1} M \cap \tilde{U}_i) = \bigcup_{i \in I} r_{U_i}^{-1} (M \cap \mathcal{M}(U_i)).$$

The next results should be compared with [Be] Sect. 1.6. They say that as far as quasi-separated spaces  $X$  with the subsequent condition  $(*)$  are concerned the theory of Berkovich is the theory of the space  $\mathcal{M}(X)$ . In the following we extend the notations  $\tilde{U}$  and  $\underline{U}$  in the obvious way to arbitrary admissible open subsets  $U \subseteq X$ .

**Proposition 4.** *Suppose that  $X$  is quasi-separated and that it has an admissible affinoid open covering  $X = \bigcup_{i \in I} U_i$  such that*

(\*) for any  $i \in I$  there are only finitely many  $j \in I$  with  $U_i \cap U_j \neq \emptyset$ ;

then  $\mathcal{M}(X)$  is a locally compact and paracompact Hausdorff space. Moreover any point in  $\mathcal{M}(X)$  has a fundamental system of compact neighbourhoods of the form  $\bar{V}$  for some quasi-compact admissible open subset  $V \subseteq X$ .

*Proof.* Let  $a, b$  denote distinct points in  $\mathcal{M}(X)$ . Since the  $\mathcal{M}(U_i)$  are Hausdorff it follows from Lemma 2 that we can choose, for any  $i \in I$ , closed subsets  $M_i, N_i \subseteq \bar{U}_i$  with union  $\bar{U}_i$  and such that  $a \notin M_i$ ,  $b \notin N_i$ . Define  $M := \bigcup_{i \in I} M_i$  and  $N := \bigcup_{i \in I} N_i$ . The condition (\*) implies that the intersections of  $M$  and  $N$  with  $\bar{U}_i$  are closed. Therefore  $M$  and  $N$  are closed in  $\mathcal{M}(X)$  by Lemma 3; their union is  $\mathcal{M}(X)$  and  $a \notin M$ ,  $b \notin N$ . This shows that  $\mathcal{M}(X)$  is Hausdorff. By a similar reasoning the union of  $\bar{U}_i$  with  $i$  running through any subset of  $I$  is closed in  $\mathcal{M}(X)$ . Applying this to those  $\bar{U}_i$  which do not contain a given point  $a \in \mathcal{M}(X)$  we see that the union of the finitely many other  $\bar{U}_i$  which do contain  $a$  is a compact neighbourhood of  $a$ . Hence  $\mathcal{M}(X)$  is locally compact. It is also clear now that the  $\bar{U}_i$  for  $i \in I$  form a locally finite covering of  $\mathcal{M}(X)$  by compact subsets. This implies that  $\mathcal{M}(X)$  is paracompact ([E] 5.1.34). The above argument showed that a point  $a \in \mathcal{M}(X)$  has at least one neighbourhood of the form  $\bar{V}$  for some quasi-compact admissible open subset  $V \subseteq X$ . Using a finite affinoid open covering of  $V$  a simple topological argument therefore implies that our second assertion only has to be checked in the case of an affinoid space  $X$ . This is done in [B] 2.2.3(iii).

Of course, in the situation of Proposition 4 the space  $\mathcal{M}(U)$ , for any admissible open subset  $U \subseteq X$ , is Hausdorff.

An abelian sheaf  $F$  on  $X$  (or  $\mathcal{P}(X)$ ) is called *overconvergent* if its restriction to any affinoid open subset  $U \subseteq X$  (or  $\mathcal{P}(U)$ ) is overconvergent. It is shown in [S] Sect. 2 that it suffices to test this condition for an admissible affinoid open covering of  $X$ . It is immediate from the affinoid case that the functor  $r_X^*$  maps any abelian sheaf on  $\mathcal{M}(X)$  to an overconvergent sheaf on  $\mathcal{P}(X)$  (or  $X$ ).

**Lemma 5.** *With the same assumptions as in Proposition 4 let  $U \subseteq X$  be a quasi-compact admissible open subset and consider the commutative diagram*

$$\begin{array}{ccc} \mathcal{P}(U) & \xrightarrow{\varphi} & \mathcal{P}(X) \\ r_U \downarrow & & \downarrow r_X \\ \mathcal{M}(U) & \xrightarrow{\psi} & \mathcal{M}(X) \end{array}$$

where  $\varphi$  and  $\psi$  are the natural maps; for any overconvergent sheaf  $S$  on  $\mathcal{P}(X)$  the base change map

$$\psi^* r_{X*} S \xrightarrow{\cong} r_{U*} \varphi^* S$$

is an isomorphism.

*Proof.* (The above diagram in general is not cartesian. The map  $\varphi$  is an open immersion whereas  $\psi$ , by Lemma 2, is a closed immersion.) It is convenient to introduce the following notation. For any two admissible open subsets  $V \subseteq W \subseteq X$  let  $S(V, W)$  denote the sheaf on  $\tilde{W}$  which is the direct image of  $S|_{\tilde{V}}$ . The proof proceeds in several steps by imposing additional assumptions which gradually will be weakened.

*Step 1.*  $U$  and  $X$  both are affinoid. Then the assertion is an immediate consequence of Theorem 4.2.

*Step 2.*  $X$  is separated,  $U$  is affinoid, and the sheaf is of the form  $S(V, X)$  for some affinoid open subset  $V \subseteq X$ . Since by [S] 2.4 with  $S$  also  $S(V, X)$  is overconvergent we may apply Step 1 to  $V \cap U \subseteq V$  and obtain

$$\begin{aligned} r_{U*}\varphi^*S(V, X) &= r_{U*}S(V \cap U, U) \\ &= \text{direct image on } \underline{U} \text{ of } r_{V \cap U*}(S|_{\tilde{V} \cap \tilde{U}}) \\ &= \text{direct image on } \underline{U} \text{ of } (r_{V*}(S|_{\tilde{V}}))|_{\underline{V} \cap \underline{U}} \\ &= \psi^*(\text{direct image on } \mathcal{M}(X) \text{ of } r_{V*}(S|_{\tilde{V}})) \\ &= \psi^*r_{X*}S(V, X). \end{aligned}$$

*Step 3.*  $X$  is separated and quasi-compact and  $U$  is affinoid. Then we have the exact sequence of sheaves

$$0 \rightarrow S \rightarrow \bigoplus_{i \in I} S(U_i, X) \rightarrow \bigoplus_{i, j \in I} S(U_i \cap U_j, X)$$

where  $I$  can be taken to be finite. We apply now Step 2 to the middle and the right hand terms and we use that the functors involved in the base change map are left exact and commute with finite direct sums.

*Step 4.*  $X$  is separated and quasi-compact. Let  $U = V_1 \cup \dots \cup V_r$  be a covering by affinoid open subsets  $V_j \subseteq U$  and let  $\varphi_j : \mathcal{P}(V_j) \rightarrow \mathcal{P}(U)$  and  $\psi_j : \mathcal{M}(V_j) \rightarrow \mathcal{M}(U)$  be the natural maps. The assertion may be checked after restricting to  $\mathcal{M}(V_j)$  for all  $1 \leq j \leq r$ . But then using Step 3 twice for  $V_j \subseteq X$  and  $V_j \subseteq U$  we obtain

$$\psi_j^*\psi^*r_{X*}S = (\psi\psi_j)^*r_{X*}S = r_{V_j*}(\varphi\varphi_j)^*S = r_{V_j*}\varphi_j^*\varphi^*S = \psi_j^*r_{U*}\varphi^*S.$$

*Step 5.* The sheaf is of the form  $\mathcal{S}(V, X)$  for some separated and quasi-compact admissible open subset  $V \subseteq X$ . Redo Step 2 but now using Step 4 instead of Step 1.

*Step 6.* In the general situation we consider the exact sequence of sheaves

$$0 \rightarrow S \rightarrow \prod_{i \in I} S(U_i, X) \rightarrow \prod_{i, j \in I} S(U_i \cap U_j, X).$$

The condition (\*) implies that the infinite products appearing in this sequence coincide with the corresponding direct sums. Therefore we can redo Step 3 now based on Step 5.

**Theorem 6.** *With the same assumptions as in Proposition 4 we have that  $r_{X*}$  and  $r_X^*$  are quasi-inverse equivalences between the category of overconvergent sheaves on  $\mathcal{P}(X)$  (or  $X$ ) and the category of all abelian sheaves on  $\mathcal{M}(X)$ .*

*Proof.* We have to show that the two adjunction maps are isomorphisms. This can be checked after restriction to  $\mathcal{P}(U_i)$  and  $\mathcal{M}(U_i)$ , respectively. But applying Lemma 5 and Theorem 4.2 we obtain

$$(r_X^* r_{X*} S)|_{\mathcal{P}(U_i)} = r_{U_i}^*((r_{X*} S)|_{\mathcal{M}(U_i)}) = r_{U_i}^* r_{U_i*}(S|_{\mathcal{P}(U_i)}) = S|_{\mathcal{P}(U_i)}$$

for any overconvergent sheaf  $S$  on  $\mathcal{P}(X)$  and similarly

$$(r_{X*} r_X^* T)|_{\mathcal{M}(U_i)} = r_{U_i*}((r_X^* T)|_{\mathcal{P}(U_i)}) = r_{U_i*} r_{U_i}^*(T|_{\mathcal{M}(U_i)}) = T|_{\mathcal{M}(U_i)}$$

for any sheaf  $T$  on  $\mathcal{M}(X)$ .

The assumptions of Proposition 4 are satisfied by any reasonable rigid space. For example, the generic fiber of any formal scheme of finite type and flat over  $k^0$  is quasi-separated and quasi-compact. To discuss the property  $(*)$  a little further we first note that obviously any morphism  $\beta : X \rightarrow Y$  of rigid spaces over  $k$  induces a continuous map

$$\begin{aligned} \mathcal{P}(\beta) : \mathcal{P}(X) &\rightarrow \mathcal{P}(Y) \\ p &\mapsto \{V \subseteq Y \text{ admissible open} : \beta^{-1}V \in p\}. \end{aligned}$$

This map respects maximal filters: To see this we may assume  $X$  and  $Y$  to be affinoid; then it is a consequence of Lemma 2.6. Hence  $\beta$  also induces a continuous map

$$\mathcal{M}(\beta) : \mathcal{M}(X) \rightarrow \mathcal{M}(Y).$$

If  $\iota : U \xrightarrow{\subseteq} X$  is the inclusion of an admissible open subset then  $\mathcal{M}(\iota)$  is injective but in general not open. We therefore introduce the following notion.

**Definition.** *An open immersion  $\iota : U \rightarrow X$  is called wide open if  $\mathcal{M}(\iota)$  is an open immersion, too.*

In case  $\iota$  is a wide open inclusion map we call  $U$  simply a wide open subset of  $X$ . By Lemma 3 this notion is local in  $X$ .

**Lemma 7.** *For any admissible open subset  $U \subseteq X$  the following conditions are equivalent:*

- i.  $U$  is wide open in  $X$ ;
- ii.  $r_X^{-1}(U) = \tilde{U}$ ;
- iii.  $\underline{U}$  is open in  $\mathcal{M}(X)$ .

*Proof.* It is trivial that i. implies iii. Assume now iii. to hold. It is clear that  $\tilde{U} \subseteq r_X^{-1}(U)$ . Let  $p \in r_X^{-1}(\underline{U})$  be any prime filter. Since  $r_X^{-1}(\underline{U})$  is open in  $\mathcal{P}(X)$  we find an affinoid open subset  $V \subseteq X$  such that  $V \in p$  and  $\underline{V} \subseteq \underline{U}$ . The latter implies that  $V \subseteq U$  and hence  $U \in p$  which means that  $p \in \tilde{U}$ .

The assertion ii. can be expressed by saying that the obvious diagram of topological spaces

$$\begin{array}{ccc} \mathcal{P}(U) & \xrightarrow{\mathcal{P}(i)} & \mathcal{P}(X) \\ r_U \downarrow & & \downarrow r_X \\ \mathcal{M}(U) & \xrightarrow{\mathcal{M}(i)} & \mathcal{M}(X) \end{array}$$

is cartesian. Moreover  $\mathcal{P}(i)$  is an open immersion and  $r_U$  and  $r_X$  are quotient maps. It is straightforward that then  $\mathcal{M}(i)$  has to be an open immersion, too.

**Proposition 8.** *With the same assumptions as in Proposition 4 let  $U \subseteq X$  be a wide open subset which possesses a countable admissible affinoid open covering; then  $U$  is quasi-separated and satisfies the condition  $(*)$  in Proposition 4.*

*Proof.* It follows from Proposition 4 that  $\mathcal{M}(X)$  is locally compact. As an open subset  $\mathcal{M}(U)$  is locally compact, too. Of course  $U$  is quasi-separated. The assumption about the countable covering then implies by Lemma 2 that  $\mathcal{M}(U)$  is a countable union of compact subsets or in other words is countable at infinity and in particular is paracompact.

According to [Bou] I.9.10 Corollary there is a locally finite open covering

$$\mathcal{M}(U) = \bigcup_{i \in J} M_i$$

such that all the  $\bar{M}_i$  are compact. Moreover, by [Bou] IX.4.3 Theorem 3 there are open coverings

$$\mathcal{M}(U) = \bigcup_{i \in J} N_i = \bigcup_{i \in J} L_i$$

such that

$$\bar{L}_i \subseteq N_i \subseteq \bar{N}_i \subseteq M_i \quad \text{for any } i \in J.$$

The  $\bar{N}_i$  and  $\bar{L}_i$  are compact as well. We claim that for any  $i \in J$  there are only finitely many  $j \in J$  with  $\bar{N}_i \cap \bar{N}_j \neq \emptyset$ . Fix an  $i \in J$ . The family  $\{\bar{N}_j : j \in J\}$  being locally finite we find, for any  $a \in \bar{N}_i$ , an open neighbourhood  $M_a$  of  $a$  in  $M_i$  which intersects only finitely many sets  $\bar{N}_j$ . Since  $\bar{N}_i$  is compact we have

$$\bar{N}_i \subseteq M_{a_1} \cup \dots \cup M_{a_r} \quad \text{for some points } a_1, \dots, a_r.$$

If now  $\bar{N}_i \cap \bar{N}_j \neq \emptyset$  then also  $M_{a_\rho} \cap \bar{N}_j \neq \emptyset$  for some  $1 \leq \rho \leq r$ . Hence this can happen only for finitely many  $j \in J$ .

According to Proposition 4 any point in some  $\bar{L}_i$  has a neighbourhood in  $N_i$  of the form  $\underline{V}$  for some quasi-compact admissible open subset  $V \subseteq U$ . Since the  $\bar{L}_i$  are compact and cover  $\mathcal{M}(U)$  it follows that there exists a family  $\{V_i\}_{i \in I}$  of affinoid open subsets in  $U$  such that

- $U = \bigcup_{i \in I} V_i$ ,
- for any  $i \in I$  there are only finitely many  $j \in I$  with  $V_i \cap V_j \neq \emptyset$ , and
- for any point  $a \in \mathcal{M}(U)$  there is an  $i \in I$  such that  $\underline{V}_i$  is a neighbourhood of  $a$  in  $\mathcal{M}(U)$ .

The latter property implies that the  $V_i$  form an admissible covering of  $U$ : Let  $\beta: Y \rightarrow U$  be any morphism from an affinoid space  $Y$  into  $U$ . Using the compactness of  $\mathcal{M}(Y)$  it easily follows that the covering  $\{\beta^{-1}V_i\}_{i \in I}$  of  $Y$  has a finite subcovering and hence is admissible.

Now let  $X = Z^{an}$  be the rigid space associated to a separated scheme  $Z$  of finite type over  $k$  ([BGR] 9.3.6). Of course  $X$  is separated.

**Lemma 9.** *If  $Z_0 \subseteq Z$  is a Zariski open subscheme then  $U := Z_0^{an}$  is wide open in  $X = Z^{an}$ .*

*Proof.* Using Lemma 3 and an admissible affinoid open covering of  $Z^{an}$  this follows from the fact that any Zariski open subset in an affinoid space is wide open ([S] Sect. 3 Proposition 3(iii)).

**Proposition 10.**  *$X = Z^{an}$  satisfies the condition  $(*)$  in Proposition 4.*

*Proof.* We want to apply Proposition 8. By Nagata  $Z$  is Zariski open in a proper scheme  $\bar{Z}$  over  $k$ . Since  $\bar{Z}^{an}$  is proper and hence quasi-compact it satisfies the assumptions of Proposition 4. According to the previous Lemma  $Z^{an}$  is wide open in  $\bar{Z}^{an}$ . By Proposition 8 it remains to check that  $Z^{an}$  has a countable admissible affinoid open covering. Writing  $Z$  as a finite union of affine open subschemes we are reduced to consider the case where  $Z$  is affine. But for the affine space and then also for any Zariski closed subscheme in the affine space our claim is obvious.

Using the properties of the Raynaud functor ([Meh] or [BL] 4.1) one can easily establish a version of Theorem 3.4 for any quasi-separated and quasi-compact space  $X$ .

## References

- [B] V.G. Berkovich, Spectral theory and analytic geometry over non-Archimedean fields, Providence 1990
- [Be] V.G. Berkovich, Etale cohomology for non-archimedean analytic spaces, Publ. Math. IHES **78** (1993), 5–161
- [BGR] S. Bosch, U. Güntzer, R. Remmert, Non-Archimedean Analysis, Berlin-Heidelberg-New York 1984
- [BL] S. Bosch, W. Lütkebohmert, Formal and rigid geometry I. Rigid spaces, Math. Ann. **295** (1993), 291–317
- [Bou] N. Bourbaki, General Topology, Berlin-Heidelberg-New York 1989
- [D] P. Deligne, Letter to Raynaud, 1992
- [E] R. Engelking, General topology, Warschau 1977
- [FP] J. Fresnel, M. van der Put, Géométrie Analytique Rigide et Applications, Progress in Math. vol. 18, 1981
- [GP] L. Gerritzen, M. van der Put, Schottky groups and Mumford curves, Springer Lect. Notes in Math. **817**, 1980
- [H] R. Huber, Continuous valuations, Math. Z. **212** (1993), 455–477
- [M] H. Matsumura, Commutative ring theory, Cambridge 1986

- [Meh] F. Mehlmann, Ein Beweis für einen Satz von Raynaud über flache Homomorphismen affinoider Algebren, *Schriftenr. Math. Univ. Münster*, 2. Ser. 19 (1981)
- [P] M. van der Put, Cohomology on affinoid spaces, *Compositio Math.* **45** (1982), 165–198
- [S] P. Schneider, Points of rigid analytic varieties, *J. reine angew. Math.* **434** (1993), 127–157